Centre for Theoretical Physics
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TENTAMEN GENERAL RELATIVITY

tuesday, 20-01-2009, rooms 5118.-152 en -156, 9.00-12.00

Indicate at the first page clearly your name, address, date of birth, year of arrival and at every other page your name.

Question 1

Consider a manifold with coordinates $\{x^a\}$ $(a = 1, \cdots, n)$.

(1.1) Suppose $T^a_b(x)$ is a tensor field of rank (1,1) on this manifold. How does this tensor field transform under a coordinate transformation $x^a \rightarrow x^{a\prime}(x)$?

(1.2) Consider a contravariant vector field $V^a(x)$. How does the derivative

$$\frac{\partial V^a(x)}{\partial x^b} \tag{1}$$

of this vector field transform under a coordinate transformation $x^a \rightarrow x^{a\prime}(x)$? Compare your result with the answer you obtained in question (1.1). Can you conclude that the derivative given in Eq. (1) is a tensor of rank (1,1)?

(1.3) Give the definition of the covariant derivative $\nabla_b V^a$ in terms of the affine connection Γ^a_{bc}. How should the covariant derivative $\nabla_b V^a$ by definition transform under a coordinate transformation $x^a \rightarrow x^{a\prime}(x)$? Is $\nabla_b V^a$ a tensor?

(1.4) Compare the answers to questions (1.2) and (1.3) and derive how the affine connection should transform under a coordinate transformation $x^a \rightarrow x^{a\prime}(x)$. Is the affine connection a tensor?

Question 2

The Newtonian limit is defined by the following assumptions:

(a) $v/c << 1$: low coordinate velocities v

(b) $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(h^2)$: weak gravitational field

(c) $\partial_0 h_{\mu\nu} << \partial_i h_{\mu\nu}$ $(i = 1, 2, 3)$: gravitational field is slowly varying in time.

Consider a particle that follows the geodesic equation

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu \dot{x}^\rho = 0\,, \qquad (2)$$

where $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$ is the 4-velocity.

(2.1) Give the expression of the connection $\Gamma^\mu_{\nu\rho}$ in terms of the metric tensor and show that in the Newtonian limit the geodesic equation (2) reduces to

$$\frac{d^2x^i}{dt^2} + \frac{1}{2}(\partial^i h_{00})\, c^2 = 0 \qquad (i = 1, 2, 3)\,. \qquad (3)$$

(2.2) Use Eq. (3) to derive a relation between Eintein's metric tensor component g_{00} and Newton's gravitational potential ϕ.

(2.3) Consider in the Newtonian limit an object with mass M. The Newtonian limit can only be trusted if we are much farther away from the object than a characteristic distance which is the so-called Schwarzschild radius r_s. Use the answer of question (2.2) to derive an expression for r_s.

Outside the object the metric is given by the Schwarzschild metric

$$ds^2 = c^2(1 - \frac{2m}{r})dt^2 - (1 - \frac{2m}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \qquad (4)$$

where m is a parameter.

(2.4) Derive an expression for the parameter m in terms of the mass M of the object. Is the Newtonian limit valid at $r = 2m$?

Question 3

Consider the Robertson-Walker metric (we take $c = 1$)

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \qquad k = +1, 0, -1, \quad (5)$$

We assume that the energy-momentum tensor of the Universe is given by that of a perfect fluid

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}, \quad (6)$$

where $\rho = \rho(t)$ is the density, $p = p(t)$ is the pressure and u_μ is the 4-velocity.

(3.1) Compare this expression with that of a gas of particles

$$T^{\mu\nu} = \sum_n p_n^\mu p_n^\nu \frac{1}{E_n} \delta^{(3)}(\vec{x} - \vec{x}_n(t)) \quad (7)$$

and show that in a co-moving frame the following inequality is satisfied:

$$0 \le p \le \frac{1}{3}\rho. \quad (8)$$

(3.2) Explain to what kind of situations the limiting cases $p = 0$ and $p = \frac{1}{3}\rho$ correspond to.

Consider the Friedmann models with $p = 0$ and zero cosmological constant. From the Einstein equations and the continuity equation of the energy-momentum tensor one can derive the following two equations:

$$\dot{\rho} + 3\rho\frac{\dot{R}}{R} = 0, \qquad \frac{\dot{R}^2 + k}{R^2} = \frac{1}{3}\kappa\rho, \qquad k = +1, 0, -1, \quad (9)$$

where κ is the gravitational coupling constant.

(3.3) Use Eqs. (9) to show that we live in a closed Universe, i.e. $k > 0$, if $\rho_0 > \rho_c$, where ρ_0 is the present-day density and ρ_c is some critical density. Derive an expression for this critical density in terms of κ and the Hubble 'constant'.

(3.4) The $k = 1$ Friedmann model is given by the following solution:

$$R(\psi) = \frac{1}{2}A^2(1 - \cos\psi)\,, \qquad\qquad t(\psi) = \frac{1}{2}A^2(\psi - \sin\psi)\,, \tag{10}$$

for some parameter ψ and constant A. What is the lifetime of this Universe? For which value of t has the Universe maximal size?